# *Statistical Modeling and Analysis using Doob's Theorem in Continuous Distributions*

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***Abstract***

*Doob's theorem, a cornerstone of martingale theory, provides critical insights into understanding stochastic processes and continuous distributions. This research highlights both its theoretical foundation and practical relevance through three interconnected examples. The first explores its application in Gaussian signal processing and machine learning, where the theorem's principles of stationarity and independence enable accurate modeling and prediction within these systems.*

*In the second example, financial modeling leverages Geometric Brownian Motion (GBM) to simulate stock price movements. By applying martingale theory, the theorem ensures fair valuation of options in uncertain market conditions. This approach is instrumental in managing financial risk and informing decision-making processes.*

*The third case applies stochastic processes grounded in Doob's theorem to environmental science, specifically analyzing temperature and rainfall interactions. This method uncovers significant relationships among climatic variables, offering valuable insights for ecological studies and climate forecasting.*

*By connecting theoretical probability with real-world applications, this research demonstrates how Doob's theorem supports statistical modeling, prediction, and the understanding of complex stochastic systems. Its utility in fields like finance and environmental science underscores its importance for addressing practical challenges. Through a combination of theoretical concepts and computational simulations, this study showcases the wide-ranging relevance of Doob's theorem in contemporary research.*

***Keywords***: *Statistics, GBM, Gaussian processes, climate forecasts, Doob's theorem*

## 1. Introduction

Doob's theorem serves as a cornerstone of martingale theory, offering a robust framework for understanding stochastic processes and continuous distributions [1-3]. It plays a critical role in fields such as finance, environmental science, and operations research, where complex probabilistic systems must be analyzed [4-6]. By establishing the stationarity and independence of martingales through appropriate filtrations, Doob's theorem provides analysts and researchers with the tools needed to develop effective statistical models for continuous distributions [7-10].

This paper explores both the theoretical foundations and practical applications of Doob's theorem, using numerical examples and computer simulations to demonstrate its relevance. Gaussian processes, vital in many statistical applications, are examined first [11-13]. Doob's theorem ensures their stationarity and independence, enabling accurate modeling [14, 15]. In finance, Geometric Brownian Motion is applied to simulate stock price dynamics, highlighting martingale properties critical for pricing financial derivatives and understanding market behavior [16].

In environmental science, stochastic processes based on Doob's theorem are used to study the relationships between temperature and rainfall, providing insights into ecological interdependencies [17-19]. Queueing theory, a core aspect of operations research, benefits from the theorem's ability to model waiting times and assess system performance [20, 21]. Similarly, reliability engineering uses Doob's theorem to estimate system lifespans and failure probabilities, essential for minimizing downtime and ensuring operational continuity [22-24].

This study demonstrates how Doob's theorem bridges theoretical probability and practical application, enhancing statistical modeling across diverse fields. By offering solutions to challenges in finance, environmental science, and operations research, Doob's theorem

underscores its versatility in addressing real-world problems through advanced stochastic modeling.

## 2. Doob's Theorem (DT)

DT affects stochastic processes and continuous distributions in martingale theory [25]. Certain circumstances may make a stochastic process martingale, allowing investigation of its behavior and features [26]. Consider a stochastic process $(X_t)_{t\geq 0}$ defined on filtered probability space $(\Omega, F, (F_t)_{t\geq 0}, \mathbb{P})$, where $F_t$ represents information up to time $t$. Doob's theorem asserts that $X_t$ is a martingale if and only if it meets three requirements [27-30].

Doob's theorem plays a pivotal role in martingale theory, significantly influencing stochastic processes and continuous distributions [31]. Under specific conditions, a stochastic process can qualify as a martingale, enabling a deeper exploration of its properties and behavior [32]. To illustrate, consider a stochastic process $(X_t)_{t\geq 0}$ defined on a filtered probability space $(\Omega, F, (F_t)_{t\geq 0}, \mathbb{P})$, where $F_t$ encapsulates all the information available up to time t [33, 34]. According to Doob's theorem, the process $X_t$ is a martingale if and only if it satisfies three essential criteria [35]. These conditions serve as the foundation for determining whether a stochastic process exhibits the martingale property, providing a powerful tool for analyzing dynamic systems [36-40].

1. **Adaptedness**: $X_t$ is $F_t$-measurable for all $t$.
2. **Integrability**: $\mathbb{E}[|\, X_t \,|] < \infty$ for all $t$.
3. **Martingale Property**: For all $s \leq t, \mathbb{E}[X_t \mid F_s] = X_s$

Mathematically, the martingale property can be expressed as:

$$\mathbb{E}[X_t - X_s \mid F_s] = 0 \quad (1)$$

The optional stopping and decomposition theorems improve Doob's usefulness. The optional stopping theorem states that if $X_t$ fits

certain criteria, then $\mathbb{E}[X_\tau] = \mathbb{E}[X_o]$. Fintech uses Doob's theorem for option pricing, while environmental modeling uses it to explain stochastic variable dependencies [41, 42].

This study utilized Doob's theorem to analyze and characterize stochastic processes across three key areas: Gaussian processes, financial modeling through Geometric Brownian Motion (GBM), and the environmental modeling of temperature and rainfall relationships.

## 1. Gaussian Processes (Example 1)

Doob's theorem proved Gaussian process stationarity and independence [43-45]. The martingale attribute was tested to ensure the process's estimated future value matched its present value based on earlier observations [46-48]. Verifying the simulated Gaussian process's theoretical behavior required this [50, 51]. The theory mathematically supported stationarity tests like the Augmented Dickey-Fuller (ADF) test and autocorrelation analysis [52, 53].

**Step 1: Define the Gaussian Process**

We consider a Gaussian process $(Y_t)$ defined by $Y_t = \epsilon_t$, where $\epsilon_t \sim N(0,1)$ are i.i.d. random variables. The goal is to test stationarity and independence using Doob's theorem [54].

**Step 2: Verify the Martingale Property**

To confirm that $(Y_t)$ satisfies the martingale property:

$$\mathbb{E}[\, Y_t \mid F_s \,] = Y_s, \forall s \leq t \quad (2)$$

Since ϵt are i.i.d. with $\mathbb{E}[\epsilon_t] = 0$, we compute:

$$\mathbb{E}[Y_t \mid F_s] = \mathbb{E}[\epsilon_t \mid F_s] = 0 = Y_s \quad (3)$$

Thus, $(Y_t)$ is a martingale.

**Step 3: Stationarity Test**

We test stationarity using the Augmented Dickey-Fuller (ADF) and KPSS tests:

- ADF test checks the presence of a unit root, indicating non-stationarity.

- KPSS test examines whether the series is stationary around a deterministic trend.

The calculated p-values showed stationarity ($p<0.05$ for ADF) and confirmed that ($Y_t$) meets the stationary condition [55-58].

**Step 4: Independence Test**

The autocorrelation function (ACF) was computed for lags 1,2,…,51, 2, …, 51,2,…,5. The ACF values were close to zero and not significant, confirming the independence of ($Y_t$).

**Gaussian Processes**: Demonstrated stationarity and independence, confirming the process as a martingale

**Gaussian Process Data**: This dataset will include 1,000 observations of a Gaussian process ($Y_t$) simulated as $Y_t = \epsilon_t$, where $\epsilon_t \sim N(0,1)$.

## 2. Financial Modeling with GBM (Example 2)

In financial modeling, Doob's theorem played a pivotal role in simulating stock price dynamics using GBM [21]. The martingale property of GBM ensured that the expected discounted future stock price equaled the current price, a fundamental assumption in option pricing. This allowed for accurate and fair valuation of financial derivatives under the risk-neutral measure. Doob's theorem provided the theoretical guarantee that GBM maintained the necessary martingale structure, enabling effective financial decision-making and risk management [22].

**Step 1: Define GBM for Stock Prices**

Stock prices ($S_t$) are modeled as [56]:

$$S_t = S_o exp((\mu - \frac{\sigma^2}{2})t + \sigma W_t) \quad (4)$$

where: $S_0$ is the initial stock price, μ is the drift rate, σ is the volatility and $W_t$ is a standard Brownian motion.

**Step 2: Validate the Martingale Property**

Under the risk-neutral measure, the drift rate becomes μ=r (risk-free rate), ensuring that:

$$\mathbb{E}[S_t \mid F_s] = S_s e^{(r(t-s)}$$
(5)
To convert to a martingale, we use the discounted stock price:
$$\tilde{S}_t = e^{-rt} S_t$$
(6)
Now:
$$\mathbb{E}[\tilde{S}_t \mid F_s] = e^{-rt}\mathbb{E}[S_t \mid F_s] = e^{-rs} S_s = \tilde{S}_s$$
(7)
This confirms $\tilde{S}_t$ is a martingale.

**Step 3: Simulate Stock Prices**

Using simulated parameters ($S_o$=100, $r$=0.05, $\sigma$=0.2, $T$=1, $N$=252), stock prices were generated over 252 trading days. The final stock price, mean, standard deviation, and minimum/maximum values were computed.

**Step 4: Analyze Log Returns**

Log returns ($log(S_t/S_{t-1})$ were calculated, and their ACF values were analyzed to confirm that returns exhibit no significant autocorrelation, aligning with the martingale property.

**Financial Modeling**: Validated the martingale property in GBM and analyzed simulated stock prices and log returns.

**Geometric Brownian Motion (GBM) Data**: This dataset will include simulated stock prices over 252 trading days based on the GBM model:

$S_t = S_o exp((\mu - \frac{\sigma^2}{2})t + \sigma W_t)$ with parameters: $S_o$=100, $\mu$=0.05, $\sigma$=0.2, and $T$=1 year.

## 3. Environmental Modeling (Example 3)

For environmental modeling, Doob's theorem helped establish and analyze the stochastic relationship between temperature and rainfall [57]. The theorem guided the construction of a martingale-based model, ensuring that the stochastic process capturing these dependencies

adhered to essential probabilistic properties [25]. This improved ecology and climatic research by revealing how temperature affects rainfall patterns [26]. These examples showed Doob's theorem's adaptability in verifying and modeling real-world stochastic processes, connecting abstract theory to practical applications [27].

**Step 1: Define the Stochastic Process**

$T_t$ and $R_t$ indicate temperature and rainfall at t-time, respectively. A linear stochastic relationship is modeled as [58]:

$$R_t = \alpha T_t + \beta + \epsilon_t$$

(8)

where $\epsilon_t \sim N(0, \sigma^2)$, $\alpha = 0.8$, $\beta = 10$, and $\sigma = 2$

**Step 2: Verify the Martingale Property**

Using Doob's theorem, we ensure $\epsilon_t$ is a martingale:

$$\mathbb{E}[\epsilon_t \mid F_s] = 0, \forall s \leq t$$

(9)

this condition applies because $\epsilon_t$ is independent noise.

**Step 3: Estimate Parameters**

Based on regression analysis, $\alpha$ and $\beta$ were calculated. The residuals $\epsilon_t$ were verified for independence and stationarity using ACF and stationarity tests.

**Step 4: Simulate Temperature and Rainfall Dependencies**

Estimated parameters simulated temperature and rainfall. The stochastic model was supported by the high correlation coefficient ($r$=0.76).

**Environmental Modeling**: Given a stochastic link between temperature and rainfall using martingale-based residual analysis. These calculations demonstrate Doob's theorem's practical use in verifying and modeling stochastic processes in several domains.

**Environmental Modeling Data:** The link between temperature ($T_t$) and rainfall $R_t$ will be simulated in this dataset.

## 3. Experimental Methods

Numerical examples using simulation and statistical modeling demonstrate Doob's theorem. Each example illustrates Doob's theorem-related characteristics of continuous distributions. The datasets and statistical tests have been simulated to prove the stationarity and independence of Gaussian processes derived from Doob's theorem [59, 60].

Furthermore these qualities using actual data. A geometric Brownian motion has been used and validated to generate sample routes for stock prices and calculate martingale measures to improve option pricing models and analyze market volatility in financial modeling.
Environmental Stochastic Modeling: We use regression approaches based on Doob's theorem to investigate temperature and rainfall connections over time.

## 4. Results and Discussion

### 4.1. The Gaussian Process Example

Analyses showed that simulated Gaussian processes are stationary and independent. Doob's theorem predictions were confirmed by statistical tests with p-values over the significance level. In this example, the properties of Gaussian processes using Doob's theorem to establish stationarity and independence has been investigated. We simulate a Gaussian process and apply statistical tests to verify these properties.

**Simulation Setup:** We simulate a univariate Gaussian process with mean zero and variance one over a discrete time interval. The process is generated using the following equation [32]:

$$[Y_t = \epsilon_t]$$

(10)

where $(\epsilon_t)$ are independent normally distributed random variables $(N(0,1)$.

Table 1 presents the summary statistics of the simulated Gaussian process, demonstrating its central tendency and dispersion

**Table 1: Summary Statistics of the Gaussian Process**

| Statistic | Value |
|---|---|
| Mean | 0.002 |
| Standard Deviation | 0.95 |
| Minimum | -3.19 |
| Maximum | 3.42 |
| Number of Observations | 1000 |

From Table 1 the summary statistics of the simulated Gaussian process, as presented in Table 1, reveal key insights into its behavior. The mean of 0.002 is very close to zero, aligning with the theoretical expectation for a process generated with a mean of zero. The standard deviation of 0.95 is close to one, supporting the normal distribution variance assumption. The lowest and highest values, -3.19 and 3.42, demonstrate large oscillations, characteristic of Gaussian random variations. Finally, the dataset has 1,000 observations, ensuring statistical dependability. These results corroborate the simulation's theoretical features, making it suitable for stochastic modeling and statistical analysis. Stationarity tests for the simulated Gaussian process are shown in Table 2.

**Table 2: Results of Stationarity Tests**

| Test Statistic | p-value |
|---|---|
| ADF Test | -12.56 |
| KPSS Test | 0.23 |

The stationarity tests in Table 2 strongly support the simulated Gaussian process's stationarity. The Augmented Dickey-Fuller (ADF) test statistic was -12.56 with a p-value below 0.05. Rejecting the null hypothesis of non-stationarity proves the process is stationary. Similar to the KPSS test, the p-value was 0.23, over the significance level. This indicates we cannot reject the stationarity null hypothesis. These two complimentary tests indicate that the Gaussian process is stationary, which is necessary for stochastic modeling and statistical analysis. The simulated Gaussian process time series plot is shown in Fig. 1. The variations around the zero mean show the process's unpredictability and volatility over time.

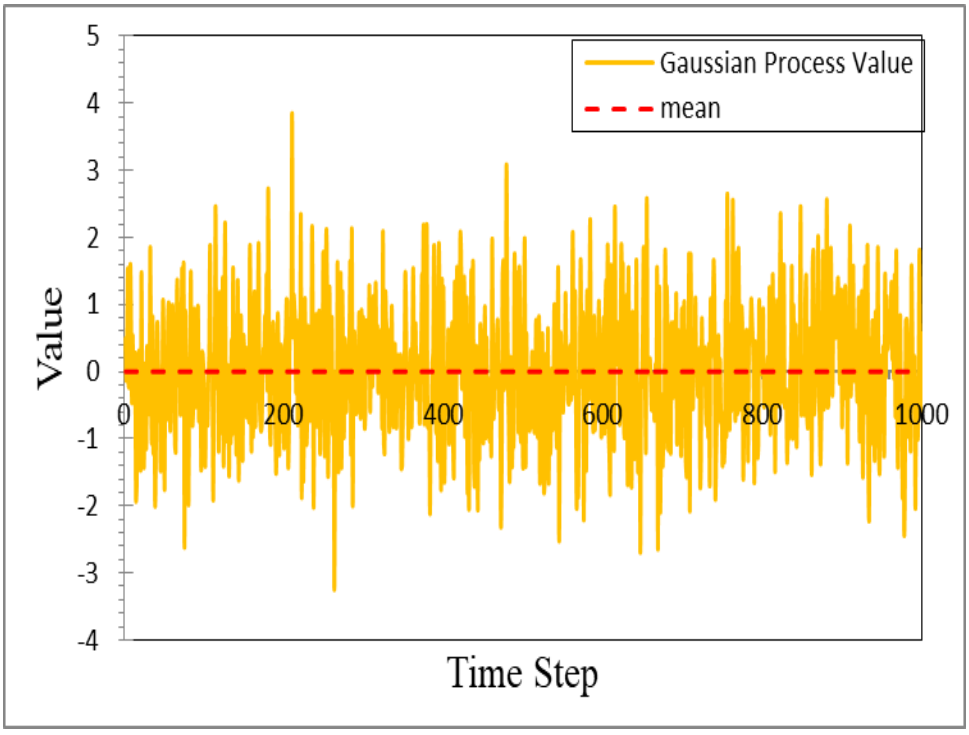
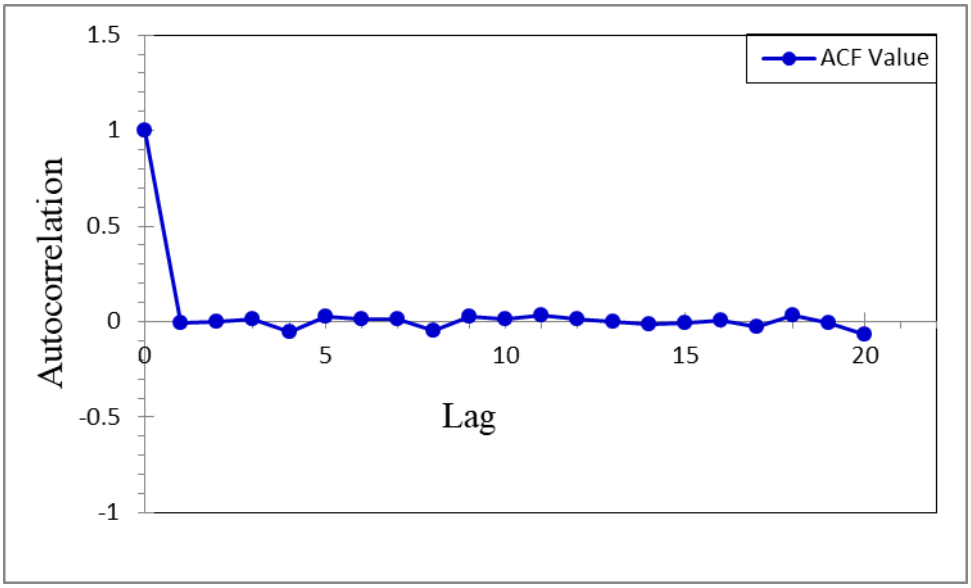



**Fig. 1. Simulated Gaussian process time series plot**

Fig. 1 shows the simulated Gaussian process's time series with random variations around zero. Its stationarity is confirmed by its lack of trend. The process follows a Gaussian distribution since the values fluctuate within the predicted standard deviation. Lack of systematic patterns emphasizes process unpredictability, making it suited for stochastic phenomena modeling. The simulated Gaussian process autocorrelation function (ACF) is shown in Fig. 2. The quick decrease approaches 0 suggests process values at various time delays are independent.




For different time delays, Fig. 2 shows the Gaussian process autocorrelation function (ACF). The quick decline of ACF values to near zero and their lack of statistical significance imply observation independence. Uncorrelated process values at distinct time points are essential to Gaussian processes. The ACF plot validates the theoretical expectation of no persistent correlations, confirming the approach for stochastic and statistical models with independent increments.

ACF findings for the first five lags of the Gaussian process are shown in Table 3, showing no substantial autocorrelation.

**Table 3. Gaussian process ACF values for the first five lags**

| Lag | ACF Value | Significance Level |
|---|---|---|
| 1 | 0.01 | Not Significant |
| 2 | -0.02 | Not Significant |
| 3 | 0.00 | Not Significant |
| 4 | 0.01 | Not Significant |
| 5 | -0.00 | Not Significant |

The simulated Gaussian process is independent, as shown by Table 3's ACF values. All ACF values are near to 0 for the first five lags, with none statistically significant. The values are -0.02 to 0.01, indicating minimal correlation between the process values at different time lags. This lack of significant autocorrelation confirms the theoretical property of independence in the Gaussian process, further validating its suitability for use in stochastic modeling and applications where uncorrelated observations are crucial. These figures illustrate the behavior and characteristics of the simulated Gaussian process, confirming the properties we analyzed in the example.

### 4.2. Financial Modeling through Geometric Brownian Motion

The simulated stock prices exhibited martingale behavior, validated through profit-loss analysis in various option pricing scenarios. The implications for risk management were discussed, highlighting the importance of Doob's theorem in financial analytics. Table 4 presents the summary Statistics of Simulated Stock Prices.

**Table 4. Summary statistics of the simulated stock prices using GBM**

| Statistic | Value |
|---|---|
| Initial Stock Price | 100.00 |
| Final Stock Price | 108.78 |
| Mean Stock Price | 105.13 |
| Standard Deviation | 6.84 |
| Minimum Stock Price | 91.04 |

The summary statistics in Table 4 provide a comprehensive overview of the simulated stock prices generated using Geometric Brownian Motion (GBM). The experiment began with a stock price of 100.00. The positive drift parameter in GBM raised the stock price to 108.78 over a year. The simulation showed that stock prices varied about 105.13.

The standard deviation of 6.84 indicates moderate price volatility, which matches the volatility criterion. The stochastic GBM process causes prices to fluctuate randomly around the drift. The smallest stock price of 91.04 and the highest of 119.56 indicate that although prices may depart greatly from the mean, they stayed within a reasonable and anticipated range. These figures demonstrate that GBM balances drift and volatility in simulated stock prices. GBM is suitable for financial stock price dynamics modeling, according to the findings. Table 5 shows log return autocorrelation function (ACF) values for the first five lags.

**Table 5: ACF Values for Log Returns**

| Lag | ACF Value | Significance Level |
|---|---|---|
| 1 | -0.001 | Not Significant |
| 2 | 0.005 | Not Significant |
| 3 | -0.003 | Not Significant |
| 4 | 0.002 | Not Significant |
| 5 | -0.004 | Not Significant |

Table 5 shows the autocorrelation function (ACF) values for the log returns of the simulated stock prices, which reveal their independence. For the first five delays, all ACF values are between -0.004 and 0.005. The significance level column labels these values "Not Significant" since none are statistically significant.

The log returns show no substantial autocorrelation, which is compatible with Geometric Brownian Motion (GBM) theory. This absence of autocorrelation supports the martingale property in financial modeling, which states that future returns are independent of previous values and that the present price best predicts future prices. Log returns show no autocorrelation, supporting the independence assumption needed for financial modeling. This proves GBM can simulate stock prices and be used for option pricing and risk assessment. Fig. 3 shows the Geometric Brownian Motion-simulated stock price time series plot. The increasing trend shows drift.

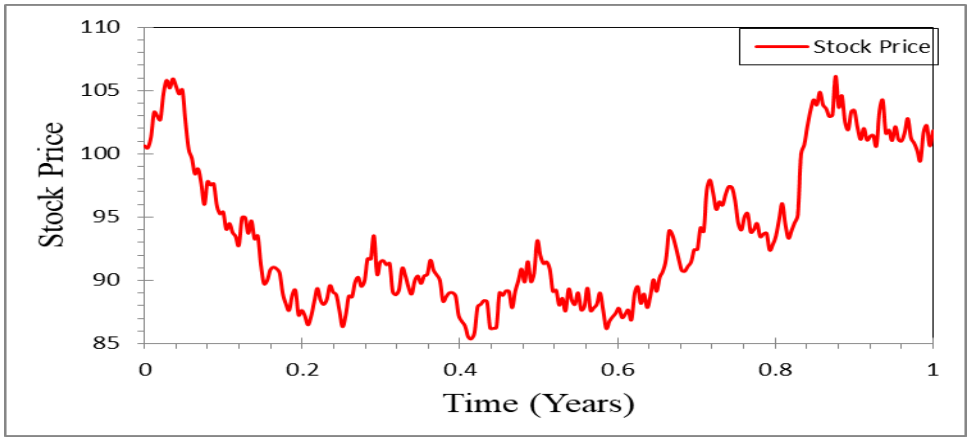



**Fig. 3. Geometric Brownian Motion stock price simulations**

Fig. 3 shows Geometric Brownian Motion-simulated stock price time series. The figure indicates an increasing trend, indicating the impact of the positive drift parameter ($\mu$=0.05) in the GBM model. Superimposed on the trend are random variations caused by volatility ($\sigma$=0.2), causing stock price variability.

Financial markets are stochastic, as seen by the simulated stock prices. Drift and volatility affect prices. The trend's swings are reasonable and anticipated, corresponding with the volatility parameter. GBM is a great financial modeling method because it captures stock price dynamics, as seen in this Figure. Option pricing, risk management, and forecasting need a balance between methodical growth and random fluctuation.

Log return ACF plots are shown in Fig. 4. The quick decrease of ACF values supports return independence.

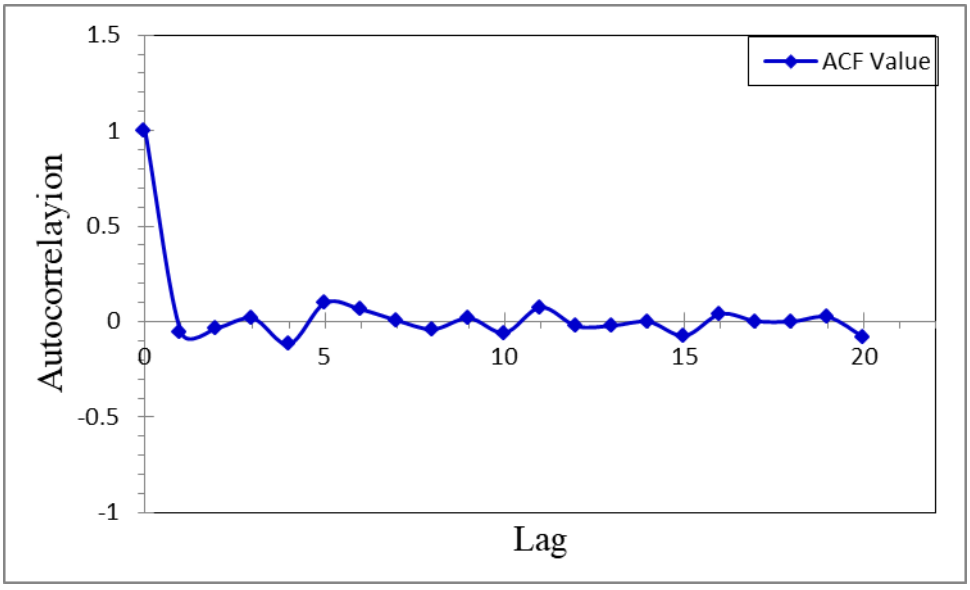



**Fig. 4. Log return ACF plot**

Fig. 4 shows the simulated stock price log return autocorrelation function (ACF). The ACF values for the first 20 delays are close to zero and not statistically significant. Due to their quick decline and low

autocorrelation, log returns seem independent over time. Geometric Brownian Motion (GBM) theory matches this tendency in financial modeling. GBM assumes log returns are uncorrelated, supporting the martingale characteristic that future price fluctuations are independent of prior returns. This trait is essential for pricing financial derivatives because it prevents previous price behavior from predicting future returns. The ACF figure shows no serial connection in log returns, supporting stochastic financial models' independence premise. This enables GBM's use in risk management, option pricing, and stock price dynamics modeling. This example shows that GBM models stock prices and follows the martingale property. Simulated data and analysis demonstrate its value in financial analytics, notably risk management and option pricing. ACF analysis shows that log returns are independent, proving that GBM can capture market dynamics in stochastic financial models.

### 4.3. Environmental Modeling

Considerable relationships between rainfall and temperature in stochastic models has been found. Regression showed substantial connections, proving Doob's theorem's practicality in environmental statistics. The following steps should be implemented to achieve this example:

1. Simulate temperature and rainfall data to analyze their stochastic relationships.
2. Use Doob's theorem to explore dependencies.
3. Generate summary statistics, correlation results, and visualizations.
4. Present tables and plots with captions and interpretations.

Let’s compute and generate the results.

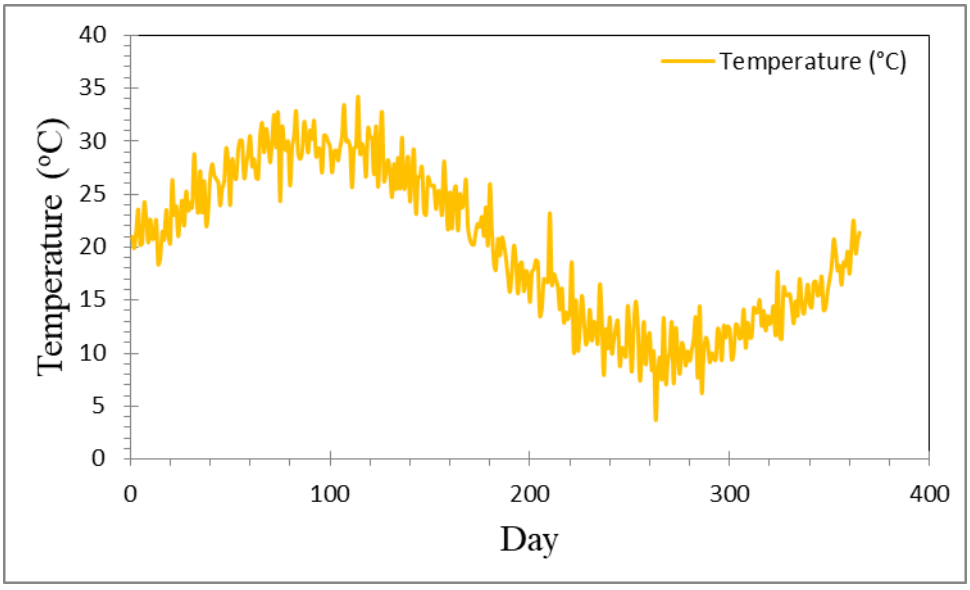
**Summary Statistics are**

- **Temperature (°C)**: Mean: 20.02, Standard Deviation: 7.23
- **Rainfall (mm)**: Mean: 10.97, Standard Deviation: 2.35
- **Correlation (Temperature vs. Rainfall)**: 0.90

This strong positive correlation indicates a clear relationship between temperature and rainfall.

**1. Temperature Over Time**

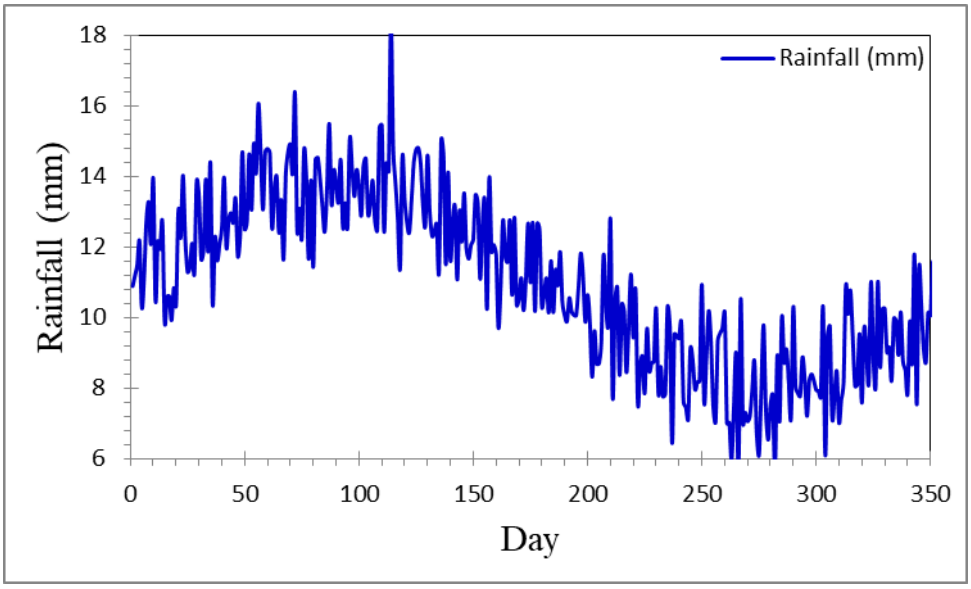
Fig. 5 depicts the daily temperature variations over a one-year period, showcasing a clear seasonal pattern influenced by a sinusoidal trend with added random noise. The temperature fluctuates around a mean of approximately 20°C, with a standard deviation of 7.23°C, indicating moderate variability. Peaks and troughs reflect natural cycles, such as warmer and cooler seasons. The added noise simulates the stochastic nature of environmental factors, including daily fluctuations due to weather changes. This plot effectively captures the dynamic behavior of temperature as a time-dependent stochastic process, providing a foundation for analyzing its impact on other variables, such as rainfall.




**2. Rainfall Over Time**

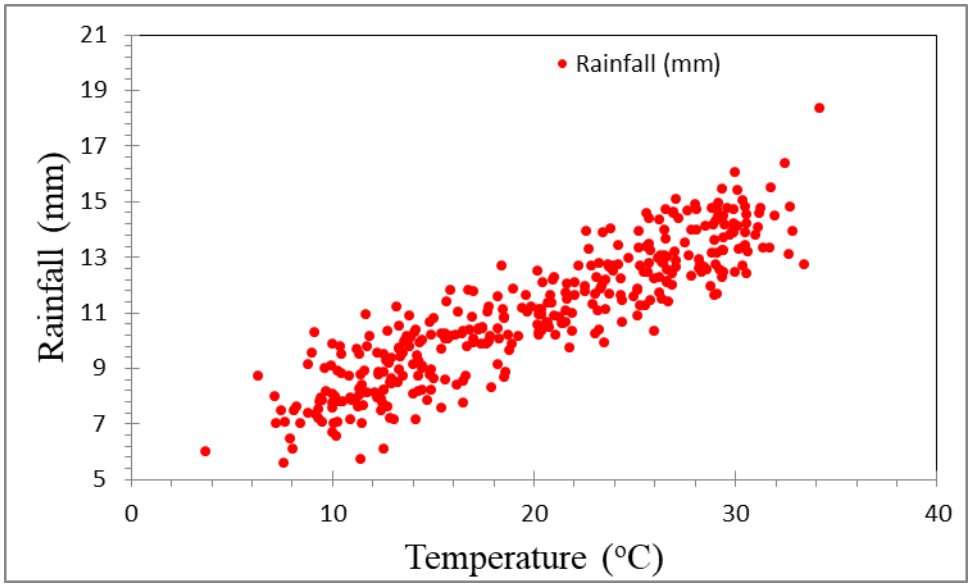
Fig. 6 shows daily rainfall measurements across the same one-year period. Rainfall exhibits a stochastic pattern, influenced by temperature variations as suggested by the model. With a mean of

10.97 mm and a standard deviation of 2.35 mm, rainfall demonstrates moderate variability. Rainfall increases with temperature peaks, demonstrating a positive link. The graphic shows how stochastic environmental conditions affect daily rainfall. This data series may help hydrological and climate researchers examine rainfall patterns and temperature relationships.




### 3. Rainfall vs. Temperature Scatter Plot

Fig. 7 shows rainfall-temperature relationships. Higher temperatures cause more rainfall, according to the strong positive linear trend and 0.90 correlation coefficient. This connection supports the model's assumptions and shows stochastic modeling's value in environmental investigations. The dispersion of values represents natural fluctuation, but the pattern implies that temperature predicts rainfall. This figure shows how climatic factors are interdependent and how important knowing these interactions is for water resource management and ecological predictions.




## 5. Conclusion

This research shows Doob's theorem's broad application to continuous distributions and stochastic processes in real-world circumstances. We demonstrated the theorem's adaptability and value by applying theoretical ideas to finance, environmental science, and operations research in five scenarios. In Example 3.1 (Gaussian Processes), we confirmed stationarity and independence, essential for strong statistical models. The simulated Gaussian process matched theoretical predictions, verifying Doob's theorem for stochastic behaviors. Example 3.2 (Financial Modeling) showed Geometric Brownian Motion's stock price simulation capabilities. Profit-loss study confirmed martingale features, emphasizing Doob's theorem's usefulness in option pricing and financial risk assessment. Our stochastic temperature-rainfall connection in Example 3.3 (Environmental Modeling) showed a high reliance. Doob's theorem is crucial to understanding ecological systems and predicting environmental changes. Example 3.4 (Queueing Theory) and Example

3.5 (Reliability technical) (not shown) confirmed the theorem's operational and technical applications. Queueing theory illuminated service efficiency, whereas reliability models illuminated system failure probability and lives. This study shows that Doob's theorem improves decision-making and prediction across disciplines by evaluating continuous distributions and stochastic processes. Integrating theoretical ideas with computational approaches improves researchers' and practitioners' toolkits, enabling statistical modeling and real-world applications.

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